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by Himmatul Mursyidah

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Himmatul Mursyidah, and Subiono

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Eigenvalue, Eigenvector, Eigenmode of Reducible Matrix and Its Application

Himmatul Mursyidah[1,a)] and Subiono[2,b)]

[1]Mathematics Education Department, Muhammadiyah University of Surabaya, Indonesia
[2]Mathematics Department, Sepuluh Nopember Institute of Technology, Surabaya, Indonesia

[a)]himmatul.pendmat@fkip.um-surabaya.ac.id
[b)]subiono2008@matematika.its.ac.id

Abstract. There are three essential components of matrix that must be understood in the process of completing the scheduling problems using max-plus algebra. They are eigenvalue, eigenvector, and eigenmode. The result showed that the reducible matrix does not necessarily have eigenvalue. If it has eigenvalue, the eigenvalue is not necessarily unique with finite value. The eigenvector corresponding to the eigenvalue of reducible matrix is not unique that contains at least a finite element. Furthermore, the eigenmode of a regular reducible matrix is not unique with all finite elements for each component.

INTRODUCTION

One of max-plus algebra application is to solve the scheduling system problem. A system can be seen as consisting of several resources that can be shared by multiple users to achieve a common goal [1]. We can determine the beginning and end of each activity on resources by using max-plus algebra, so we can do synchronization and concurrency for multiple resources on the system.

Max-plus algebra application relates to a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{D})$, where $\mathcal{N}$ is a set of all nodes and $\mathcal{D}$ is a set of some ordered (not necessarily different) pairs of nodes, called arc. In a system, resources can be seen as nodes, and the process of moving from one resource to another resource can be viewed as an arc. To perform synchronization and concurrency resources, it is necessary to determine the length of each process. Furthermore, the length of each process can be called as weight of the arc. There are two types of graphs in max-plus algebra, they are strongly connected graph and not strongly connected graph. A Graph called strongly connected, if every nodes communicate each other. Matrix representation of a strongly connected graph called irreducible matrix [2]. Meanwhile, when there is a node that is not communicate to the other node, the graph is not strongly connected and its matrix representation is called reducible matrix.

In the completing process of the scheduling problems using max-plus algebra, there are essential components relate to the matrix. They are eigenvalue, eigenvector, and eigenmode. Eigenvalue represents the periodicity of a system. Eigenvector represents the beginning time for any resources on a system. While eigenmode shows periodic behavior of a system.

So far, eigenvalue, eigenvector, especially eigenmode research in max-plus algebra much focused on the algorithm to obtain these components. It can be found in [3] and [4] which both of them discuss the generalized eigenmode algorithm for regular reducible matrix in max-plus algebra. While for implementing these algorithms, need to understand the character of the three components according to the type of matrix representation first.

Based on the role importance of eigenvalue, eigenvector, and eigenmode, and the research that had been done, so we discuss the characterization of eigenvalues, eigenvectors, and eigenmode of the reducible matrix in this paper. After we get the three components characterization, then we give an example of the result application in the queue problem.

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MAX-PLUS ALGEBRA

In this section, we give some concepts related to max-plus algebra, they are definition, operation, and the properties of max-plus algebra, graph in max-plus algebra, eigenvalues, eigenvectors, and eigenmode in max-plus algebra, and algorithms to determine those three components.

Definition 1 *[5] Max-plus algebra is non empty set $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{\varepsilon\}$ where $\mathbb{R}$ is the set of real numbers, $\varepsilon := -\infty$, and two binary operations are defined as follows:*

1. *the operation $\oplus$, that is $\forall x, y \in \mathbb{R}_\varepsilon$ satisfy*
$$x \oplus y := \max\{x, y\},$$
2. *the operation $\otimes$, that is $\forall x, y \in \mathbb{R}_\varepsilon$ satisfy*
$$x \otimes y := x + y.$$

Max-plus algebra denoted by $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ or simply written $\mathbb{R}_{\max}$ is semiring [see 1]. The set of $n \times m$ matrices on $\mathbb{R}_{\max}$ denoted by $\mathbb{R}_{\max}^{n\times m}$. A matrix in $\mathbb{R}_{\max}^{n\times m}$ is called regular if it contains at least one element different from ε in each row. There are two forms of matrix in $\mathbb{R}_{\max}^{n\times m}$ that need to be understood, they are matrix $\mathcal{E}(n, m)$ and $E(n, m)$. Matrix $\mathcal{E}(n, m)$ is $n \times m$ matrix where all elements equal to ε. While matrix $E(n, m)$ is $n \times m$ matrix where

$$[E(n,m)]_{i,j} := \begin{cases} e & \text{untuk} \quad i = j, \\ \varepsilon & \text{untuk} \quad i \neq j. \end{cases}$$

and $e = 0$. If $m = n$, then E is a square matrix and called the identity matrix.

Elements of $\mathbb{R}_{\max}^n := \mathbb{R}_{\max}^{n\times 1}$ called a vector. The j-th element of a vector $\mathbf{x} \in \mathbb{R}_{\max}^n$ denoted by x_j or can be written as $[\mathbf{x}]_j$. A vector in $\mathbb{R}_{\max}^n$ where all elements equal to e is called a unit vector, and denoted by $\mathbf{u}$.

Graph in Max-Plus Algebra

Max-plus algebra application is closely related to the directed graph denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{D})$, where $\mathcal{N}$ is a set of all nodes and $\mathcal{D}$ is a set of some ordered (not necessarily different) pairs of nodes. Ordered pair of nodes in a graph called arc. Therefore, if nodes $i, j \in \mathcal{N}$ then the arc from i to j denoted by (j, i). A graph can be represented into a square matrix in max-plus algebra. A corresponding graph to a square matrix A on $\mathbb{R}_{\max}$ is called a communication graph, and denoted by $\mathcal{G}(A)$. Suppose $A \in \mathbb{R}_{\max}^{n\times n}$, element $a_{i,j} \neq \varepsilon$ if there is an arc from node j to node i in $\mathcal{G}(A)$.

The path term in graph is defined as an arc sequence that represents the link between a node with another node. A path is called elementary if no nodes appear more than once in it. A circuit is a closed elementary path, that has an initial node as same as final node. Suppose a path p, the weight of it denoted by $|p|_w$ is the sum of weights for each arc in the path p. The length of path p is the number of arcs in it, which is denoted by $|p|_\ell$. The average weight of path p is the weight divided by the length of path.

Average circuit is an average weight of a circuit. A circuit that has a maximum average circuit called critical circuit. Critical graph of $\mathcal{G}(A)$, denoted by $\mathcal{G}^c(A) = (\mathcal{N}^c(A), \mathcal{D}^c(A))$ is a graph consisting of a set of nodes and arcs in critical circuits of $\mathcal{G}(A)$. Graph and its matrix representation are related each other. A graph condition can be known according to the matrix representation, and vice versa. For example, the path length in a graph associated with the power of its matrix representation.

Theorem 2 *[2] Let $A \in \mathbb{R}_{\max}^{n\times n}$. It holds for all $k \geq 1$ that*

$$[A^{\otimes k}]_{i,j} = \max\{|p|_w : p \in P(j, i; k)\},$$

where $[A^{\otimes k}]_{i,j} = \varepsilon$ in the case where $P(j, i; k)$ is empty, i.e., when no path of length k from i to j exists in $\mathcal{G}(A)$.

Furthermore, we can define A^+ matrix from Theorem 2, i.e.,

$$A^+ := \bigoplus_{k=1}^{\infty} A^{\otimes k}.$$

Matrix $A^{\otimes k}$ stop for $k = n$. It is given in the following theorem.

Theorem 3 *[1] Let $A \in \mathbb{R}_{\max}^{n\times n}$ be such that any circuit in $\mathcal{G}(A)$ has average circuit weight less than or equal to e. Then, it holds that*

$$A^+ := A \oplus A^{\otimes 2} \oplus \ldots \oplus A^{\otimes n} \in \mathbb{R}_{\max}^{n\times n}.$$

There are two types of graphs based on the connectedness property, they are strongly and not strongly connected graph. The main topic discussed in this paper is not strongly connected graph. Before we discuss the definition both types of graph, we must understand the reachable and communicate terms first. For any two nodes $i, j \in \mathcal{N}$, node i is said to be reachable from node j, denoted by $j\mathcal{R}i$, if there exist a path from i to j. While the node i communicates with node j, denoted by $j\mathcal{C}i$, if and only if $i = j$ or node i is reachable from the node j and node j is reachable from node i. Relation $\mathcal{C}$ is equivalence relation in $\mathcal{N}$.

A graph is called strongly connected if all nodes in the graph communicate with each other another i.e., for every $i, j \in \mathcal{N}$ satisfy $i\mathcal{C}j$. A matrix in $\mathbb{R}_{\max}^{n\times n}$ representation of a strongly connected graph is called irreducible matrix. Furthermore, irreducible matrix is a matrix that can not be constructed into upper triangular form.

If there is a node that does not communicate to the other nodes, the graph is called not strongly connected. A matrix in $\mathbb{R}_{\max}^{n\times n}$ representation of not strongly connected graph is called reducible matrix. Furthermore, reducible matrix is a matrix that can be constructed into a block upper triangular matrix form. A block upper triangular matrix consists of matrices $\mathcal{E}$ and irreducible matrices, and it is representation of reduced graph. The reduced graph is a reduction result of not strongly connected graph.

Eigenvalue, Eigenvector, and Eigenmode in Max-Plus Algebra

In the max-plus algebra application, there are three essential components associated with the matrix. They are eigenvalue, eigenvector, and eigenmode.

Definition 4 *[2] Let $A \in \mathbb{R}_{\max}^{n\times n}$ be a square matrix. If $\mu \in \mathbb{R}_{\max}$ is a scalar and $\mathbf{v} \in \mathbb{R}_{\max}^{n}$ is a vector that contains at least one finite element such that*

$$A \otimes \mathbf{v} = \mu \otimes \mathbf{v},$$

then μ is called an eigenvalue of A and $\mathbf{v}$ an eigenvector of A associated with eigenvalue μ.

Suppose $C(A)$ is a set of all circuits in $\mathcal{G}(A)$. The maximum average circuit, denoted by λ is defined as

$$\lambda := \max_{p \in C(A)} \frac{|p|_w}{|p|_\ell}.$$

If the maximum average of graph $\mathcal{G}(A)$ i.e., λ has a finite value, then it is the eigenvalues of matrix A. While the column $[A_\lambda^*]_{.,\eta}$ is an eigenvector of A assosiated with eigenvalues λ. The statement is according with the following lemma.

Lemma 5 *[2] Let the communication graph $\mathcal{G}(A)$ of matrix $A \in \mathbb{R}_{\max}^{n\times n}$ have finite maximal average circuit weight λ. Then, the scalar λ is an eigenvalue of A, and the column $[A_\lambda^*]_{.,\eta}$ is an eigenvector of A assosiated with λ, for any node $\eta \in \mathcal{G}^c(A)$.*

In the process of resolving scheduling problem, closely related to efforts getting sequence $\{\mathbf{x}(k) : k \in \mathbb{N}\}$ of this linear equation model

$$\mathbf{x}(k+1) = A \otimes \mathbf{x}(k), \tag{1}$$

for $k \geq 0$, where $A \in \mathbb{R}_{\max}^{n\times n}$ and $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}_{\max}^{n}$ is the initial state.

The definition of generalized eigenmode is given in Definition 6.

Definition 6 *[2] A pair of vectors $(\eta, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$ is called a generalized eigenmode of the regular matrix A if for all $k \geq 0$*

$$A \otimes (k \times \eta + \mathbf{v}) = (k+1) \times \eta + \mathbf{v}.$$

We need to know that a generalized eigenmode is also called eigenmode.

Algorithm for Determining The Eigenvalue, Eigenvector, and Eigenmode in Max-Plus Algebra

In this paper, we use the *power algorithm* and the generalized eigenmode algorithm for reducible regular matrices.

Algorithm 7 *Power Algorithm [1]*

1. *Take an arbitrary initial vector* $\mathbf{x}(0) \neq \mathbf{u}[\varepsilon]$.
2. *Iterate equation (1) until there are integers* $p > q \geq 0$ *and a real number c, such that a periodic regime is reached i.e.,*

$$\mathbf{x}(p) = c \otimes \mathbf{x}(q).$$

3. *Compute as the eigenvalue*

$$\lambda = \frac{c}{p-q}.$$

4. *Compute as the eigenvector*

$$\mathbf{v} = \bigoplus_{i=1}^{p-q} \left(\lambda^{\otimes(p-q-i)} \otimes \mathbf{x}(q+i-1) \right).$$

Algorithm 8 *Generalized Eigenmode Algorithm for Regular Reducible Matrices [3]*

1. *Take a regular reducible matrix* $A \in \mathbb{R}_{\max}^{n \times n}$.
2. *Specify a block upper triangular matrix form of matrix A.*
3. *Compute eigenvalue and eigenvector of the last block matrix in the main diagonal of block upper triangular matrix. Suppose* $A_{q,q}$*, compute eigenvalue* $\lambda_q = \lambda(A_{q,q})$ *and eigenvector* $\mathbf{v}_q$ *associated with the eigenvalue. Then, take* $\xi_q = \lambda_q$ *and* $i = q$.
4. *Compute eigenvalue* $\lambda_{(i-1)}$ *of matrix* $A_{(i-1),(i-1)}$.
5. *If* $\lambda_{(i-1)} > \xi_i$ *go to 6. If not, go to 7.*
6. *Set* $\xi_{(i-1)} = \lambda_{(i-1)}$ *and compute vector* $\mathbf{v}_{(i-1)}$ *according to this following equation:*

$$\xi_{(i-1)} \otimes \mathbf{v}_{(i-1)} = A_{(i-1),(i-1)} \otimes \mathbf{v}_{(i-1)} \oplus \bigoplus_{j=i}^{q} A_{(i-1),j} \otimes \mathbf{v}_j.$$

   Then, go to 8.
7. *Set* $\xi_{(i-1)} = \lambda_i$ *and compute vector* $\mathbf{v}_{(i-1)}$ *according to this following equation:*

$$\lambda_i \otimes \mathbf{v}_{(i-1)} = A_{(i-1),(i-1)} \otimes \mathbf{v}_{(i-1)} \oplus \bigoplus_{j=i}^{q} A_{(i-1),j} \otimes \mathbf{v}_j.$$

   Then, go to 8.
8. *If* $i-1 \neq 1$ *go back to 4, if not finish.*

EIGENVALUE, EIGENVECTOR, AND EIGENMODE OF REDUCIBLE MATRIX IN MAX-PLUS ALGEBRA

In this section, we discuss about eigenvalue, eigenvector, and eigenmode characterization of reducible matrix. Then, we use the characterization result to solve a problem. The problem that we have learned is the queue problem.

To obtain the eigenvalue and eigenvector of a matrix $A \in \mathbb{R}_{\max}^{n \times n}$, we use the algorithm repeatedly of this linear equation

$$\mathbf{x}(k+1) = A \otimes \mathbf{x}(k), \ k = 0, 1, 2, \ldots. \tag{2}$$

The periodic regime of equation (2) for reducible matrix A closely related to cycle time vector, i.e.,

$$\lim_{k \to \infty} \frac{\mathbf{x}(k)}{k}.$$

The limit is exist for all $\mathbf{x}(0) \neq \mathbf{u}(\varepsilon)$, where $\mathbf{u}(\varepsilon) = \begin{pmatrix} \varepsilon & \varepsilon & \dots & \varepsilon \end{pmatrix}^T$.

If the matrix A in the equation (2) is reducible matrix, then we can always establish A be a block upper triangular matrix:

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \dots & \dots & A_{1,q} \\ \mathcal{E} & A_{2,2} & \dots & \dots & A_{2,q} \\ \mathcal{E} & \mathcal{E} & A_{3,3} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \dots & \mathcal{E} & A_{q,q} \end{pmatrix}, \tag{3}$$

Matrix $A_{i,i}$ is irreducible matrix or $A_{i,i} = \varepsilon$ for all $i \in \underline{q}$.

Eigenvalue, Eigenvector, and Eigenmode Characterization of Reducible Matrix

Some of reducible matrix that represent not strongly connected graph have eigenvalues and some are not. Here is an existence eigenvalues and eigenvectors theorem of reducible matrix.

Theorem 9 *[1] Let $\mathbf{x}(0) \neq \mathbf{u}[\varepsilon]$ be an arbitrary initial condition. If the equation system (2) satisfies $\mathbf{x}(p) = c \otimes \mathbf{x}(q)$ for several integers p, q with $p > q \geq 0$ and real number c, then*

$$\lim_{k \to \infty} \frac{\mathbf{x}(k)}{k} = \begin{pmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{pmatrix},$$

where $\lambda = \frac{c}{p-q}$. Furthermore, λ is an eigenvalue of matrix A with the eigenvector

$$\mathbf{v} = \bigoplus_{i=1}^{p-q} \left(\lambda^{\otimes(p-q-i)} \otimes \mathbf{x}(q+i-1) \right).$$

A reducible matrix does not necessarily have a unique eigenvalue. The following example shows the eigenvalue of reducible matrix is not unique.

Example 10 *Given a reducible matrix represents not strongly connected graph $\mathcal{G}(A)$ as follows:*

$$A = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 3 \end{pmatrix}.$$

Eigenvalue of matrix A is not unique. It is clear from the following description:

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 3 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} = 1 \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 3 \end{pmatrix} \otimes \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon \\ 3 \end{pmatrix} = 3 \otimes \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}.$$

We can conclude that 1 and 3 are eigenvalues of matrix A.

Furthermore, we also give an example that a reducible matrix has a unique eigenvalue.

Example 11 *Given a reducible matrix represents not strongly connected graph $\mathcal{G}(B)$ as follows:*

$$B = \begin{pmatrix} 1 & 0 \\ \varepsilon & 0 \end{pmatrix}.$$

Eigenvalue of matrix B is unique. It is clear from the following description:

$$\begin{pmatrix} 1 & 0 \\ \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} = 1 \otimes \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}.$$

While

$$\begin{pmatrix} 1 & 0 \\ \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} a \\ 0 \end{pmatrix} = \lambda \otimes \begin{pmatrix} a \\ 0 \end{pmatrix},$$

we get

$$\max\{1 + a, 0\} = \lambda + a, \tag{4}$$

and

$$\max\{\varepsilon, 0\} = \lambda. \tag{5}$$

From equation (5) we get $\lambda = 0$, and if it is substituted to the equation (4),

$$\max\{1 + a, 0\} = a. \tag{6}$$

So, we can not find a that satisfies equation (6).

Eigenvector of reducible matrix is not unique because some eigenvectors of reducible matrix can be formed by operation $\otimes$ any scalar real number with an eigenvector.

Theorem 12 *For all reducible matrices $A \in \mathbb{R}_{\max}^{n\times n}$ that have eigenvalue, having not unique eigenvector. If $\mathbf{v} \in \mathbb{R}_{\max}^{n}$ is an eigenvector associated with eigenvalue λ, then $\alpha \otimes \mathbf{v}$ is also eigenvector associated with eigenvalue λ for any $\alpha \in \mathbb{R}$.*

Proof Let λ as eigenvalue of reducible matrix A and $\mathbf{v} \in \mathbb{R}_{\max}^{n}$ is an eigenvector associated with eigenvalue λ such that

$$A \otimes \mathbf{v} = \lambda \otimes \mathbf{v}. \tag{7}$$

Multiply any scalar $\alpha \in \mathbb{R}$ on both sides of the equation (7),

$$\alpha \otimes A \otimes \mathbf{v} = \alpha \otimes \lambda \otimes \mathbf{v}. \tag{8}$$

Operation $\otimes$ is commutative, so the equation (8) becomes

$$\begin{aligned} A \otimes \alpha \otimes \mathbf{v} &= \lambda \otimes \alpha \otimes \mathbf{v} \\ A \otimes (\alpha \otimes \mathbf{v}) &= \lambda \otimes (\alpha \otimes \mathbf{v}). \end{aligned}$$

From the description, we also get $\alpha \otimes \mathbf{v} \in \mathbb{R}_{\max}^{n}$ as eigenvector of matrix A associated with eigenvalue λ, and we can conclude that the eigenvector of reducible matrix is not unique. ■

Eigenvalue of reducible matrix has finite value. The statement is given in this following theorem.

Theorem 13 *If reducible matrix $A \in \mathbb{R}_{\max}^{n\times n}$ has eigenvalue, then the eigenvalue has finite value element of real number.*

Proof According to the Theorem 9, if a reducible matrix $A \in \mathbb{R}_{\max}^{n\times n}$ has eigenvalue, it means the equation system (2) satisfies $\mathbf{x}(p) = c \otimes \mathbf{x}(q)$ for several integers $p > q \geq 0$ and real number c such that we get eigenvalue of A i.e.,

$$\lambda(A) = \frac{c}{p - q}.$$

Since c is real number and $p - q$ is integer, then we get the eigenvalue of reducible matrix A is finite real number. Moreover, if we take the distinct eigenvalue belongs to one of the block matrix in the main diagonal of upper triangular block matrix, then obviously the eigenvalue has finite value because the matrix blocks in the main diagonal are irreducible that have finite value for its eigenvalue, or the matrix blocks are ε that do not have eigenvalue. ■

While eigenvector of reducible matrix based on Example 10 and Definition 4, obtained that the eigenvector of reducible matrix contains at least one finite element.

The next discussion is about eigenmode characterization of regular reducible matrix. Before we discuss about the eigenmode existence of regular reducible matrix, first we discuss about solution of equation $\mathbf{x} = (A \otimes \mathbf{x}) \oplus \mathbf{b}$ and inhomogeneous recurrent equations.

Solution of equation $\mathbf{x} = (A \otimes \mathbf{x}) \oplus \mathbf{b}$ is given in the following theorem.

Theorem 14 *[1] Let $A \in \mathbb{R}_{\max}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}_{\max}^{n}$. If the average weight of circuit in the graph $\mathcal{G}(A)$ less than or equal 0, then $\mathbf{x} = A^* \otimes \mathbf{b}$ with $A^* := E \oplus A^+ = \bigoplus_{i=0}^{\infty} A^{\otimes i}$ is solution of $\mathbf{x} = (A \otimes \mathbf{x}) \oplus \mathbf{b}$. Furthermore, if the circuit weight in $\mathcal{G}(A)$ is negative, then the solution is unique.*

The inhomogeneous recurrent equation is an extension of the linear recurrent equation $\mathbf{x}(k+1) = A \otimes \mathbf{x}(k)$. The following theorem concerning inhomogeneous recurrent equation.

Theorem 15 *[3] Consider the inhomogeneous recurrent equation*

$$\mathbf{x}(k+1) = A \otimes \mathbf{x}(k) \oplus \bigoplus_{j=1}^{m} B_j \otimes \mathbf{u}_j(k), \tag{9}$$

with $A \in \mathbb{R}_{\max}^{n \times n}$ irreducible matrix with eigenvalue λ, or $A \in \mathbb{R}_{\max}$ i.e., $A = \varepsilon$ with $\lambda = \varepsilon$, matrix $B_j \in \mathbb{R}_{\max}^{n \times m_j}$ with $m_j \geq 1$ satisfies $B_j \neq \mathcal{E}$, while $\mathbf{u}_j(k) \in \mathbb{R}_{\max}^{m_j}$ satisfies $\mathbf{u}_j(k) = \tau_j^k \otimes \mathbf{w}_j(k)$, $k \geq 0$ with $\mathbf{w}_j \in \mathbb{R}_{\max}^{m_j}$ and $\tau_j \in \mathbb{R}$. For some $\tau = \bigoplus_{j \in \underline{m}} \tau_j$, there exists an integer $K \geq 0$ and vector $\mathbf{v} \in \mathbb{R}^n$ such that the sequence $\mathbf{x}(k) = \mu^{\otimes k} \otimes \mathbf{v}$, with $\mu = \lambda \otimes \tau$ satisfies the recurrent equation (9) for all $k \geq K$.

In the previous discussion, it is known that the reducible matrix A can be presented in an upper triangular block matrix form, with the matrix block $A_{i,i}$ is irreducible matrix, so $\lambda_i = \lambda(A_{i,i})$ or $A_{i,i} = \varepsilon$, and we get $\lambda_i = \varepsilon$. Then, suppose we take vector $\mathbf{x}(k)$ correspondence to an upper triangular block matrix 3, i.e.,

$$\mathbf{x}(k) = \begin{pmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \\ \vdots \\ \mathbf{x}_q(k) \end{pmatrix}.$$

The upper triangular block matrix of reducible matrix A satisfy recurrent equation (9), i.e.,

$$\mathbf{x}_i(k+1) = A_{i,i} \otimes \mathbf{x}_i(k) \oplus \bigoplus_{j=i+1}^{q} A_{i,j} \otimes \mathbf{x}_j(k); \; i \in \underline{q}, \; k \geq 0. \tag{10}$$

In particular, equation (10) becomes $\mathbf{x}_q(k+1) = A_{q,q} \otimes \mathbf{x}_q(k)$ for $i = q$. Assuming A is a regular matrix, then $A_{q,q}$ is also regular. So, $A_{q,q} \neq \mathcal{E}$, and the maximal strongly connected subgraph correspondence with matrix $A_{q,q}$ has non empty arc set, as a result $A_{q,q}$ is an irreducible matrix. Therefore, there is a vector with all the finite element $\mathbf{v}_q$ and a scalar $\xi_q \in \mathbb{R}$ such that

$$\mathbf{x}_q(k) = \xi_q^{\otimes k} \otimes \mathbf{v}_q$$

satisfies $\mathbf{x}_q(k+1) = A_{q,q} \otimes \mathbf{x}_q(k)$ for all $k \geq 0$. In this case, $\mathbf{v}_q$ is the eigenvector of matrix $A_{q,q}$ correspondence with eigenvalues $\lambda_q = \lambda(A_{q,q})$, where $\xi_q = \lambda_q$.

Then, for $i \in \underline{q}$ is generally given in the following theorem.

Theorem 16 *[3] Consider the recurrent equation given by equation (10). Assume that $A_{q,q}$ is irreducible and that for $i \in \underline{q-1}$ either $A_{i,i}$ is an irreducible matrix or is equal to ε. Assume also, that the $A_{i,i}$ matrices are different from $\mathcal{E}$ for $i, \overline{j = i+1}; \; i \in \underline{q}$. Then there exist finite vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ of suitable size and scalar $\xi_1, \xi_2, \ldots, \xi_q \in \mathbb{R}$ such that the sequances:*

$$\mathbf{x}_i(k) = \xi_i^{\otimes k} \otimes \mathbf{v}_i, \; i \in \underline{q}$$

satisfy equation (10) for all $k \geq 0$. The scalars $\xi_1, \xi_2, \ldots, \xi_q \in \mathbb{R}$ are determined by:

$$\xi_i = \bigoplus_{j \in \mathcal{H}_i} \xi_j \oplus \lambda_i,$$

where $\mathcal{H}_i = \{j \in \underline{q} : j > i, A_{i,j} \neq \mathcal{E}\}$.

Theorem 16 yields a result which indicates the eigenmode existence of regular reducible matrix.

Corollary 17 *[3] Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a regular reducible matrix, then there exist a pair of vectors $(\eta, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$, a generalized eigenmode, such that for all $k \geq 0$:*

$$A \otimes (k \times \eta + \mathbf{v}) = (k+1) \times \eta + \mathbf{v}.$$

Eigenmode of regular educible matrix is not unique. It is caused the vector $\mathbf{v}$ is not unique.

Theorem 18 *For all regular reducible matrix $A \in \mathbb{R}_{\max}^{n \times n}$ have non unique eigenmode, i.e., if $(\eta, \mathbf{v})$ is eigenmode of matrix A, then $(\eta, \alpha \otimes \mathbf{v})$ is also eigenmode of matrix A where $\alpha \in \mathbb{R}$.*

Proof Suppose $(\eta, \mathbf{v})$ is eigenmode of regular reducible matrix $A \in \mathbb{R}_{\max}^{n \times n}$, then for all $k \geq 0$ η and $\mathbf{v}$ satisfy

$$A \otimes (k \times \eta + \mathbf{v}) = (k+1) \times \eta + \mathbf{v}. \tag{11}$$

Multiply any scalar number $\alpha \in \mathbb{R}$ to both sides equation (11)

$$\alpha \otimes A \otimes (k \times \eta + \mathbf{v}) = \alpha \otimes (k+1) \times \eta + \mathbf{v}. \tag{12}$$

Since the operation $\otimes$ is commutative, then equation (12) becomes

$$\begin{aligned} A \otimes (\alpha \otimes k \times \eta + \mathbf{v}) &= (k+1) \times \eta + \alpha \otimes \mathbf{v} \\ A \otimes (k \times \eta + (\alpha \otimes \mathbf{v})) &= (k+1) \times \eta + (\alpha \otimes \mathbf{v}). \end{aligned}$$

We get $(\eta, \alpha \otimes \mathbf{v})$ is also eigenmode of regular reducible matrix A for any $\alpha \in \mathbb{R}$. ■

The last discussion in the characterization of regular reducible matrix is about the value of vector elements in the eigenmode. Eigenmode of regular reducible matrix has finite elements for each component of the vector.

Theorem 19 *For all regular reducible matrix $A \in \mathbb{R}_{\max}^{n \times n}$ have eigenmode i.e., pair of vectors with all vector elements are finite.*

Proof Suppose the upper triangular block matrix (3) is the construction of reducible matrix A. Eigenmode of upper triangular block matrix construction results from the reducible matrix A be a pair of vectors $(\eta, \mathbf{v})$. It was explained that the vector

$$\eta = \begin{pmatrix} \mathbf{u}^T[\xi_1] & \mathbf{u}^T[\xi_2] & \ldots & \mathbf{u}^T[\xi_q] \end{pmatrix}^T$$

is cycle time vector, where ξ_i, $i \in \underline{q}$ is real numbers by Theorem 16. So that the vector η consists of finite elements. Then, vector $\mathbf{v}$ consists of vectors $\mathbf{v}_i$, $i \in \underline{q}$. According to Theorem 16, all elements of the vector $\mathbf{v}_i$, $i \in \underline{q}$ are finite, i.e., real numbers. So that the vector $\mathbf{v}$ only contains finite element. Then we can conclude eigenmode of reducible matrix is couple of vectors with all finite elements. ■

Characterization Result Application in The Queue System Problem

In the end of the discussion, we give an example the application of eigenvalue, eigenvector, and eigenmode characterization result in the queue system problems, i.e., in the queue system of the replacement saving types servicing process in a bank for a customer service officer.

We will analyze the queue system of the replacement saving types servicing process in a customer service officer of bank. Then for the customer service process, start from customer comes to the bank, serviced by customer service officers, until leave the bank are given in a Petri net form, Figure 1. Based on Figure 1, The Petri net of replacement saving types servicing process in a customer service officer consists of seven transitions and sixth places.

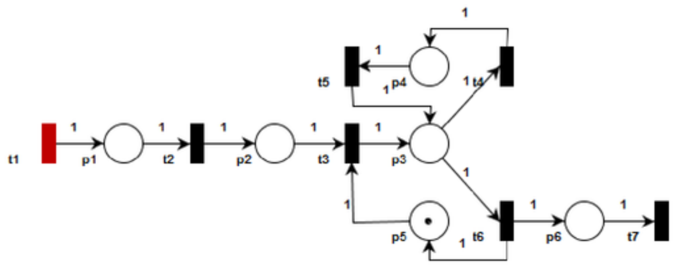

FIGURE 1. The Petri Net of Replacement Saving Types Servicing Process in A Customer Service Officer

The description of seven transitions is given as follows:

t_1 : a customer comes to the bank,
t_2 : a customer takes the queue number,
t_3 : a costumer is serviced by a customer service officer,
t_4 : a customer service officer carries the customer files to the teller,
t_5 : customer files have been processed by the teller,
t_6 : a customer has been served by customer service officer,
t_7 : a customer leaves the bank,

and six places, i.e.,

p_1 : a customer who was waiting to take a queue number,
p_2 : a customer who was waiting to be served by a customer service officer,
p_3 : a customer being served by a customer service officer,
p_4 : a customer who was waiting for files processing by the teller,
p_5 : idle or a customer services officer who is not busy,
p_6 : a customer who has been served by a customer service officer.

Furthermore, we also give the definition of the variables used in the modeling process. The variables that show the time as follows:

$t_1(k)$: time of the kth customer arrival,
$t_2(k)$: time of the kth customer taking a queue number,
$t_3(k)$: time of the kth customer begin to be served by a customer service officer,
$t_4(k)$: time of the kth customer service officer carries the customer files to the teller,
$t_5(k)$: time of the kth customer files have been processed by the teller,
$t_6(k)$: time of the kth customer has been served,
$t_7(k)$: time of the kth customer leaves the bank.

Then, the variables that determine the length of time i.e.,

$v_{t_1,k}$: the time length of the kth customer arrival,
$v_{t_2,k}$: the time length of the kth customer taking a queue number,
$v_{t_5,k}$: the time length of the kth files processing by the teller,
$v_{t_6,k}$: the time length of the kth customer served by the customer service officer,
$v_{t_7,k}$: the time length of the kth customer leaves the bank.

TABLE 1. The List of Replacement Saving Types Servicing Process in A Bank.

Codes	**Processes**	**The Length of Time** (minutes)
$v_{t_1,k}$	The time length of the kth customer arrival	8
$v_{t_2,k}$	The time length of the kth customer taking a queue number	0,5
$v_{t_5,k}$	The time length of the kth files processing by the teller	5
$v_{t_6,k}$	The time length of the kth customer served by the customer service officer	20

Based on petri net Figure 1, we obtain a model of the queue system of the replacement saving types servicing process in a customer service officer as follows:

$$\begin{pmatrix} t_1(k) \\ t_5(k) \\ t_6(k) \end{pmatrix} = \begin{pmatrix} v_{t_1,k} & \varepsilon & \varepsilon \\ v_{t_5,k} \otimes v_{t_2,k} \otimes v_{t_1,k} & v_{t_5,k} & v_{t_5,k} \\ v_{t_6,k} \otimes v_{t_5,k} \otimes v_{t_2,k} \otimes v_{t_1,k} & v_{t_6,k} \otimes v_{t_5,k} & v_{t_6,k} \otimes v_{t_5,k} \end{pmatrix} \otimes \begin{pmatrix} t_1(k-1) \\ t_5(k-1) \\ t_6(k-1) \end{pmatrix}.$$

If the time length of each process is given in Table 1, we obtain equation:

$$\begin{pmatrix} t_1(k) \\ t_5(k) \\ t_6(k) \end{pmatrix} = \begin{pmatrix} 8 & \varepsilon & \varepsilon \\ 13,5 & 5 & 5 \\ 33,5 & 25 & 25 \end{pmatrix} \otimes \begin{pmatrix} t_1(k-1) \\ t_5(k-1) \\ t_6(k-1) \end{pmatrix}.$$

The matrix of the model obtained is a reducible matrix. Furthermore, we will analyze eigenvalue, eigenvector, and eigenmode of the matrix

$$B = \begin{pmatrix} 8 & \varepsilon & \varepsilon \\ 13,5 & 5 & 5 \\ 33,5 & 25 & 25 \end{pmatrix}.$$

Based on the eigenvalue and eigenvector of reducible matrix characterization result, we get that reducible matrix does not necessarily have eigenvalue. Therefore, we use the power algorithm to find eigenvalue of reducible matrix B. With an arbitrary initial state $\mathbf{x}(0) \neq \mathbf{u}(\varepsilon)$, we can not find integers $p > q \geq 0$ and real number c that satisfy $\mathbf{x}(p) = c \otimes \mathbf{x}(q)$. Thus B has no eigenvalues. Nevertheless, since B is regular reducible matrix, then we can search eigenmode i.e., pair of vectors with finite element.

First, we must determine an upper triangular block matrix form of regular reducible matrix B to get eigenmode i.e.,

$$A = \begin{pmatrix} 5 & 5 & 13,5 \\ 25 & 25 & 33,5 \\ \varepsilon & \varepsilon & 8 \end{pmatrix}.$$

Then, we compute the eigenvalue of matrix $A_{2,2}$ i.e., $\lambda_2 = 8$, so we can take $\xi_2 = \lambda_2 = 8$ and suppose we take $\mathbf{v}_2 = 0$. The next step, we compute eigenvalue of matrix $A_{1,1} = \begin{pmatrix} 5 & 5 \\ 25 & 25 \end{pmatrix}$ by the power algorithm. With an arbitrary initial state $\mathbf{x}(0) \neq \mathbf{u}(\varepsilon)$, we get the eigenvalue of matrix $A_{1,1}$ is $\lambda_1 = 25$. Since $\lambda_1 > \xi_2$, then $\xi_1 = \lambda_1 = 25$ and we compute vector $\mathbf{v}_1$:

$$\begin{aligned} \xi_1 \otimes \mathbf{v}_1 &= (A_{1,1} \otimes \mathbf{v}_1) \oplus (A_{1,2} \otimes \mathbf{v}_2) \\ 25 \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \left(\begin{pmatrix} 5 & 5 \\ 25 & 25 \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \oplus \left(\begin{pmatrix} 13,5 \\ 33,5 \end{pmatrix} \otimes 0 \right) \end{aligned} \tag{13}$$

Based on equation (13), we get $\mathbf{v}_1 = \begin{pmatrix} -11,5 \\ 8,5 \end{pmatrix}$. Therefore, a pair of vector $(\eta, \mathbf{v})$ where $\eta = \begin{pmatrix} 25 \\ 25 \\ 8 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -11,5 \\ 8,5 \\ 0 \end{pmatrix}$ is eigenmode of matrix A because for $k = 0$, the vectors satisfy:

$$A \otimes (0 \times \eta + \mathbf{v}) = \begin{pmatrix} 13,5 \\ 33,5 \\ 8 \end{pmatrix} = 1 \times \eta + \mathbf{v},$$

for $k = 1$, satisfy:

$$A \otimes (1 \times \eta + \mathbf{v}) = \begin{pmatrix} 38,5 \\ 58,5 \\ 16 \end{pmatrix} = 2 \times \eta + \mathbf{v},$$

and so on, vectors η and $\mathbf{v}$ satisfy

$$A \otimes (k \times \eta + \mathbf{v}) = (k+1) \times \eta + \mathbf{v}.$$

for $k = 0, 1, 2, \ldots$.

Based on the eigenmode result, we can determine the time of kth customer completed in each service process. Suppose the earliest time is 08.00, then for k equal to 0 and 1 we get the result as shown in the Table 2.

TABLE 2. The Time of The First and Second Customer Service Process

Descriptions	**Time of Customer**	
	First	**Second**
Customer files have been processed by the teller	08:05:30	08:30:30
A Customer has been served by customer service officer	08:25:30	08:50:30
customer arrival	08:00:00	08:08:00

CONCLUSIONS

Based on the eigenvalue, eigenvector, and eigenmode characterization of reducible matrix, we obtain that reducible matrix does not necessarily have eigenvalue. If the reducible matrix has eigenvalue, the eigenvalue is not necessarily unique and has a finite value. Eigenvector corresponding to the eigenvalue of reducible matrix is not unique, and contains at least a finite element. Then, Regular reducible matrix does not have a unique eigenmode, with all elements are finite.

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